# MATHEMATICAL PROBLEM SOLVING PRACTICES OF HIGH SCHOOL CHILDREN: FACTORS INFLUENCING CHOICES OF STRATEGIES

Pingping Zhang
The Ohio State University
zhang.726@osu.edu

Azita Manouchehri
The Ohio State University
manouchehri.1@ous.edu

*In this work we investigated the problem solving behaviors of 3 highschool students as each solved four common non-routine problems with the goal to trace performance constancy across different subject areas and problem types. Additionally, we aimed to identify possible factors that influenced children's choices of heuristics in different problem contexts. The results suggested the individual's confidence and preference for the use of certain strategies. Inconsistency in the same individual's mathematics problem solving behaviors across different subject areas was revealed.*

## Introduction

The development of problem-solving ability among school children has been a persistent goal of mathematics education community for over a century; however, the issue of how to develop problem solving skills among learners continues to be a major dilemma. This is, in part, due to lack of specific knowledge about mathematical problem solving practices of children and factors that influence their choices and actions (English, 2010). Indeed, previous research studies on problem solving have primarily focused on effective implementation of problem solving instruction by examining students' problem solving performance on tasks (Anderson & White, 2004). These studies have identified some key factors for the success or failure of implementation of problem solving approaches in mathematics teaching; however they do not provide detailed accounts of individuals' problem solving behaviors. Muir, Beswick, and Willamson (2008) suggested that researchers must focus on understanding what successful problem solvers do and use that knowledge to help individuals develop their problem solving skills. They further argued that instead of focusing on whether particular strategies should be taught or not and how, greater attention must be devoted to understanding processes that individuals use when engaged in problem solving. In support of this suggestion we argue that knowledge about children's problem solving behaviors and factors that influence their mathematical practices while solving problems can better assist teachers in helping nurture mature problem solvers. Such knowledge is currently not well developed. The goal of the research we report here was to address this need.

The purpose of this study was to examine mathematical problems solving practices of three students in an attempt to determine whether the individuals' performances were consistent across different subject areas and problem types. Moreover, we were interested in identifying those factors that influenced children's choices of heuristics used in different problem contexts. Lastly, we intended to isolate common and unique patterns of behaviors that children exhibited as well as those factors that seemingly contributed to the institution of those patterns.

## Context and Background Literature

Nearly two decades ago Lester (1994) summarized the research community's perspectives on qualities that distinguish successful problem solvers from those characterized as poor problem solvers, and concluded that good problem solvers: know more and their knowledge is well connected and composed of rich schemata, focus more on structural features instead of literal

*Wiest, L. R., & Lamberg, T. (Eds.). (2011). Proceedings of the 33rd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Reno, NV: University of Nevada, Reno.*

features of problems, are more aware of their own strengths and weakness in terms of problem solving, monitor and regulate their problem-solving efforts more routinely and, are more concerned about obtaining best solutions to problems. More recently however, English and Sriraman (2010) argued for a reconsideration of this list, indicating that since previous research had focused mainly on solving word problems emphasized in school textbooks, primarily routine and procedural then results concerning quality of problem solving and nature of problem solving performance of children should be more critically examined. The authors attributed the lack of success of school based practices for fostering problem solving skills among children to community's inadequate knowledge about *how* individuals come to make decisions about when, where, why, and how to use heuristics and strategies when faced with novel problem contexts. Naturally, focusing on applying these strategies, without understanding how and why individuals make decisions about pathways for solving problems is non-productive (English, Lesh, & Fennewald, 2008). Despite these criticisms several fundamental factors regarding effective mathematical problem solving have been identified. First, knowledge of heuristics and their appropriate use are recognized as fundamental to mathematical problem solving (Schoenfeld, 1992). There is evidence indicating that students' use of heuristic strategies is positively related to success in problem solving, although the effect may not always be significant (Kantowski, 1977). Yet a number of studies have shown the deficiencies that students exhibit when applying heuristics and metacognitive strategies to their problem solving processes (Schoenfeld, 1992).

Flexibility in strategy use has also been referenced as a key aspect of successful problem solving. Flexibility refers to the quantity of variations that can be introduced by an individual in the concepts and mental operations one already possess (Demetriou, 2004). Elia, Heuvel-Panhuizen, and Kolovou (2009) discussed two methods for studying strategy flexibility usage: inter-task flexibility (changing strategies across problems) and intra-task flexibility (changing strategies within problems). They used three non-routine problems to study the strategy use and strategy flexibility by 4$^{th}$ grade high achievers. An implicative statistical method was performed to determine whether the strategies used by students to solve the three problems were successful or not. Guess-and-check strategy was found to be the most crucial strategy that led to the success of the three pattern/algebra problems. An important finding was that higher inter-task strategy flexibility was displayed by more successful problem solvers, while intra-task strategy flexibility did not support the problem solvers in reaching a correct answer. An intra-task strategy flexibility study showed that the understanding to the problem influenced the correctness of the answer, instead of the flexibility of the strategies.

A study of four 6$^{th}$ grade students' problem solving behaviors was conducted by Muir, Beswick, and Williamson (2008). The strategies students used in solving 6 problems were analyzed and three categories of performance were proposed to associate with the levels of problem solving behaviors students exhibited including, naive, routine and sophisticated. The consistency of approaches across problems for each individual was also studied, and the conclusion was that most individuals consistently exhibited behaviors characteristic in one category. Since all 6 problems used in this research concerned number and number sense, the consistency in performance across different content areas was not revealed. Our goal in the current study was to contribute to the existing literature on mathematical problem solving by examining the mathematical problem solving practices of 3 children on 4 tasks selected from different content areas in order to identify ways in which children's choices of heuristics and orientation may have influenced their mathematical problem solving performance.

## Methodology

A task-based interview methodology was used to closely observe and study three students as they worked on non-routine mathematical tasks. A case study report was developed for each student, describing, in detail, their actions during interview sessions. These case study reports were used, first, to identify and analyze the processes students used and patterns of problem solving behaviors they exhibited while solving problems; second, to describe and analyze the problem solving strategies children utilized, their choice of representations and metacognitive behaviors they accessed and used.

Although the major research project from which the data for the study was selected involved interviews with nearly 60 children, only three participants, Jazzy in $8^{th}$ grade, Liza and Yoni in $9^{th}$ grade (all pseudo names), were selected to serve as case subjects for the current study. This selection was due to several important considerations as described below.

First, all three participants had signed consent forms to participate in the longitudinal research project. Second, all three had worked on the same four tasks used as data collection sources. This would allow us to draw inferences regarding comparisons among their thinking and orientations, processes they used during problem solving episodes and metacognitive behaviors they exhibited. Additionally, the three participants offered a wide range of backgrounds and habits that would strengthen the potential for generalizability of the results. Despite their differences, the participants shared similar attributes; they were characterized as "successful" students of mathematics as measured by grades they had secured in their mathematics courses and results of the State mandated standardized exams. Lastly, each of the subjects displayed distinct behaviors when interacting with problems during the enrichment sessions: Liza exhibited flexible explorations when tackling problems, Jazzy showed strong reliance on calculator and numbers, and Yoni possessed the most sophisticated mathematics technique learned in school curriculum. Diverse performances were expected among the three participants.

## Data Sources

The data sources consisted of two interviews with each of the participants. Each interview consisted of approximately 35-40 minutes. Each interview was tape–recorded and used in analysis. The first interview consisted of two parts: the first part was assessing participants' mathematics background information, their beliefs about mathematics, and their views on value of mathematics for their lives. The second part of the first interview the children contained problem solving episodes. During the second interview it was reassured that the participants solved the remaining problem selected for data analysis.

The participants were interviewed individually. The protocol for problem solving interviews suggested the least interruption from interviewers during students' problem solving work. The protocol also suggested that interventions be made only when a clear understanding what children were doing was not evident or if reasoning and justification was not shared by them. The children were not restricted by a specific amount of time or their representational systems they could use.

Four non-routine problems were selected to access participants' problem solving performances in patterns, functions, and geometry (Table 1 illustrates two of the interview questions). The problems were selected so to elicit different heuristics. The diversity of subject areas and heuristics served the aim of studying the consistency of individual problem solving behaviors/performances across problem types and subject matter contexts.

1. Joe gives Nick and Tom as much money as each already has. Then Nick gives Joe and Tom as much money as each of them then has. If at the end each has 8 dollars, how much money did each have at the beginning?
4. What relationships exist among the areas of triangles and rectangle?

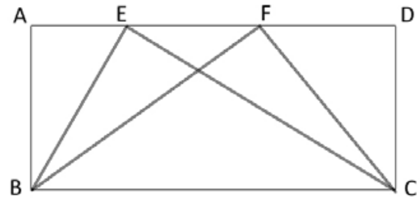


**Table 1. Description of problems**

## Data Analysis

Using Mason's mathematical problem solving model, mathematical practices of each child were analyzed and modeled. Mason (1985) divided the process of problem solving into three phases: entry, attack, and review. The entry phase includes thinking about "what do I know," "what do I want," and "what can I introduce." The review phase further contains "checking," "reflecting," and "extending." However, a typical problem solving activity is seldom linear; an individual always goes back and forth when proceeding to the desired outcome. Also it is possible that the attack phase is difficult to observe or the review phase is missing. Generally speaking, this perspective could provide an overview of the entire problem solving process so that a clearer relationship between steps could be identified and studied. Mason's model, however, does not indicate the internal or external forces that impact the movement from one phase of process to next. Between each process, distinct motives/stimuli might exist, initiated by the individual or outside information that can either facilitate or prevent making progress towards a more general understanding and ultimately, more efficient problem solving performance. Mason's model as used in this research was modified in order to demonstrate certain internal (self-initiated) and external (interviewer-initiated) forces influencing mathematical work of the children.

Each interview episode was videotaped. We first reviewed each episode repetitively to track the distinct problem solving phases of each of the children on each of the tasks during each problem solving episode. A problem by problem performance model was developed for each child and then a cross problem performance analysis was completed.

A detailed description of the processes each child used, strategies they employed and metacognitive behaviors they exhibited was completed. Every strategy and representation used by the participants was identified and recorded. Following the completion of each case study analysis, the three cases served as data sources for the overall analysis.

## Results

Table 2 provides an overview of particular behaviors and performances of children as they relate to average amount of time they spent on tasks, average number of instances of self-initiated questions, average number of times they switched strategies, average number of self-initiated testing and justifying episodes. As the data illustrate, although the three participants varied greatly in the average amount of time they spent on tasks, their performance along the self-initiated constitutive elements of the problem solving process was consistent, suggesting potential patterns of thinking and actions. The average numbers of self-initiated questions and times shifts in strategy usage had the least variety of items, indicating a less diverse use of these

two behaviors. The average numbers of justifying answers, which included self-initiated justification and interviewer-initiated justification, varied more than the previous two items. The average number of scaffolding questions that the interviewers asked was the item with the most variety, which was also the least natural item for participants. A detailed cross case analysis of subjects' mathematical practices is offered in the following section.

| | Average length of PS episode | Average number of self initiated questions | Average number of shifts in strategy usage | Average number of justifying episodes | Average number of interviewers' scaffolding questions |
|---|---|---|---|---|---|
| Liza | 10'54" | 0 | 1 | 1.5 | 1.75 |
| Jazzy | 9'57" | 0.5 | 1 | 0.75 | 5.25 |
| Yoni | 7'57" | 0.5 | 0.5 | 0.25 | 0.5 |

**Table 2. Overview of participants' particular behaviors and performances**

*Patterns of Mathematical Problem Solving Performance among Children*

Based on each student's problem solving episodes, Liza, Jazzy, and Yoni's general problem solving practices (despite content and heuristics) are illustrated in Figures 1, 2, and 3 respectively.

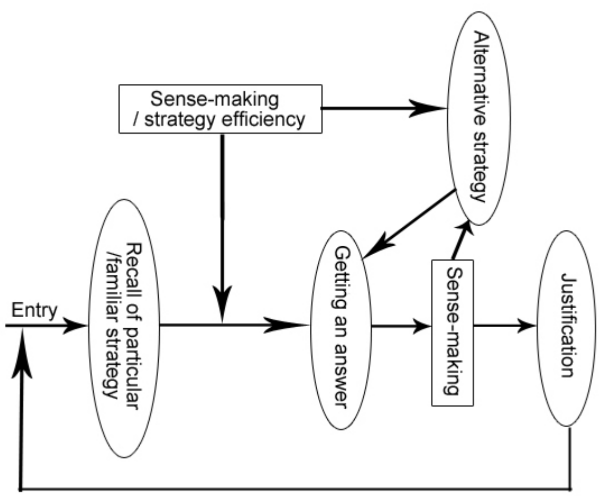


**Figure 1: Liza's general problem solving orientation**

Compared to the other two students, Liza's general problem solving process had a unique feature: sense-making. Sense-making was her way of self-monitoring, which was presented throughout all problem solving episodes. After she reached an answer, sense-making was her premier way to justify her response. It was one of the factors that could influence her switch of strategy. Liza had a belief that she could solve most problems eventually, thus she was more likely to deliberately re–enter the problem in order to gain a better understanding of the problem.

Jazzy's problem entry phase was different from other students. She always started her activity with manipulating numbers in order to get a sense of the problem. Her intra-task strategy usage depended largely on whether the strategy she used was by numerical or non-numerical. She tended to switch strategy instead of modifying information when she was not using a numerical strategy.

Yoni's general problem solving orientation was to stick with one method regardless of success or failure. He seldom justified his answers and assumed a problem was solved when he arrived as an answer (either right or wrong). His low intra-task strategy flexibility could be due to his self-confidence regarding his mathematical ability.

All three participants showed a high degree of involvement in solving the assigned problems during the interviews. They took ownership of the tasks and attempted to attack, review and solve them. The intensity of their involvement in tasks was reduced or diminished if they failed to see patterns, fully understand the information given, or make connections between what they knew and the contexts under study. An additional inhibitive factor included their ability to define mechanisms for gauging their own success in solving problems when using strategies most familiar to them.

The children's ability to identify relevant from irrelevant data, either embedded in the problem or deduced as the result of their own work, was a pivotal influence on their successful problem solving as evidenced by their willingness to reflect on options or reconsider approaches. Preoccupation with using familiar techniques learned in school curriculum served as a primary motive for participants' reluctance to focus on extending their understanding of the problems, reflecting systemically on what was given, or to even test and justify their answers.

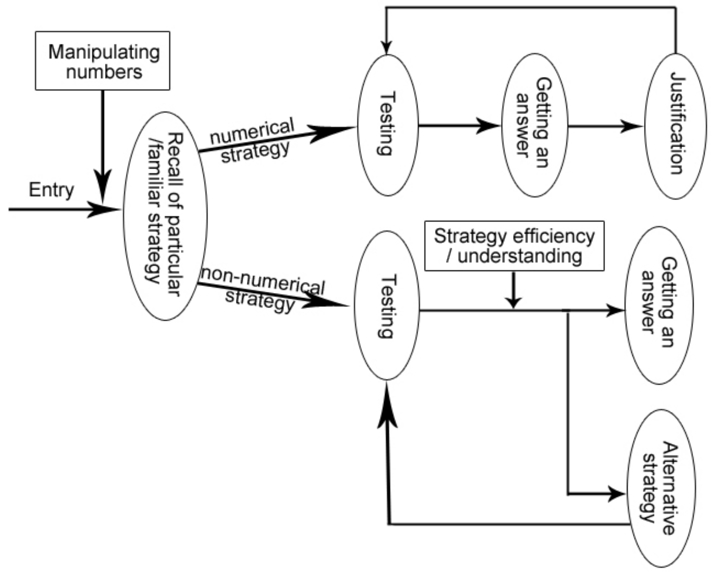


**Figure 2: Jazzy's general problem solving orientation**

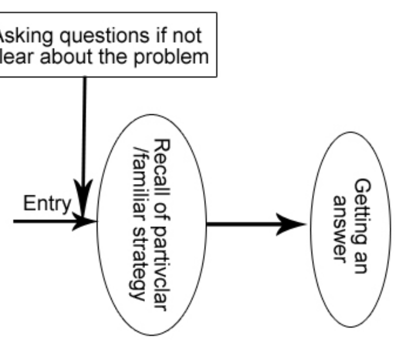


**Figure 3: Yoni's general problem solving orientation**

The children's particular orientation and their personalities influenced how they entered the problems during the initial phase, the degree of persistence they showed in solving them, and whether they tried to access additional strategies or engaged in metacognitive actions. The one subject with the least amount of interest in school mathematics and its content seemed most flexible in changing strategies. Her need for understanding and sense-making, as articulated during both interview sessions, may have been the primary force behind her natural desire to constantly examine the context at hand and to switch her approaches. On the other hand, the most academically successful student among the three, and the one with most sophisticated mathematical tools, appeared least flexible in his thinking and choices. Indeed, his attempts to use procedures he had learned in school prohibited him from taking the initiative to justify or verify his own answers or monitor his progress reflectively.

All three participants showed the tendency to enter problems using the technique of testing numerical values. They made, or refused to make modifications to their initial choices of numerical values to understand the problem better. Once, and if a deeper understanding of the problem was achieved, they were more willing to switch strategies. Most notably, they were also

*Wiest, L. R., & Lamberg, T. (Eds.). (2011). Proceedings of the 33rd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Reno, NV: University of Nevada, Reno.*

more successful in the use of newly adopted approaches. Despite this, those with greater control over numerical manipulation tended to remain loyal to the use of this approach. A shift from one strategy to another was the result of either a significant change in their level of understanding of the problem, or provoked by interviewers' questions.

All three subjects showed the tendency to use concepts and procedures most recently addressed in school at the time of data collection, regardless of whether these concepts were relevant to the problem under study. This was most notable when they worked on the geometry task. With the exception of Jazzy, the two other participants experienced difficulty when abstractions of specific knowledge became a focus of work.

The children's ability to access different representational modes was also driven by the contexts they had previously experienced in school experiences. The use of drawing a picture for illustrating the problem became only natural for two of the participants (Jazzy and Liza) when their initial attempt at using numerical data for answering questions seemed too cumbersome to be practical. Even when they were successful in use of the strategy they remained skeptical of the accuracy of their own responses. Formalizing and authenticating the final answer derived using this approach was endorsed to an outside authority (the interviewer), as opposed to self conviction.

## Discussion

The results of this work provide additional evidence suggesting that self-monitoring is positively correlated with success in performance on certain mathematical activities (Cohors-Fressenborg, Sjuts, & Sommer, 2004; Cohors–Fressenborg et al., 2010). Liza used sense-making as a way of self-monitoring her progress on tasks and towards gauging her problem solving process accordingly. Jazzy used numerical computation as a way to self-regulating her actions and increasing her control over tasks, performed better. Yoni, who did not exhibit self-monitoring/regulating consistently during his problem solving processes, was not always successful in solving problems. Hence, we highlight that self-monitoring/regulating could be a significant influence on successful problem solving on both routine and non-routine tasks, consistent with findings of previous.

Our results also indicate that intra-task strategy flexibility usage does not imply success at reaching correct answers on tasks (Elia, Heuvel-Panhuizen, & Kolovou, 2009). However, we posit further that the level of intra-task strategy flexibility usage might depend largely on the individual's level of confidence and preference for the use of certain strategies. These constructs may not ensure that correct answers across different subject areas and heuristics might be reached. Instead, they may prevent the individuals from moving forward in securing an enhanced level of understanding of the problem.

Our data also revealed inconsistency in the same individual's mathematics problem solving behaviors across different subject areas and/or heuristics usage. This result is distinct from the conclusion of previous research that indicates most individuals exhibit consistent problem solving behaviors (Muir, Beswick, & Williamson, 2008). The factors that may impact the consistency in behaviors include the preference for the use of specific approaches and orientations (visual, graphical, pictorial, etc.), experience with specific subject area (number theory, algebra, geometry), familiarity with the heuristic needed to solve the problem, and personal belief about one's own mathematical ability and confidence in problem solving.

Personal orientation could largely impact one's problem solving behaviors throughout the entire process (i.e. Jazzy's numerical orientation). Personal belief about one's own knowledge and ability could impact one's confidence: Liza believed she could eventually solve all problems

and Yoni believed he was good at mathematics; both of them exhibited noticeable confidence during their problem solving episodes. On the other hand, Jazzy's belief about problem solving ("the only right answer") influenced her attitude during the problem solving process: she asked for the right answer even when she was convinced by visual evidence.

Lastly, perhaps a puzzling finding of the study is the relationship between children's claimed level of confidence with mathematics and their problem solving performance. In virtually all past literature focused on the connections between affect and problem solving performance of children, the conclusion had been drawn that confidence and success in problem solving are directly related: the more confident an individual was in his/her mathematical ability, the better performance on problem solving was observed. Our research provides conflicting results. As described earlier, the most mathematically confident individual in this study, Yoni, showed the lowest degree of flexibility in thinking or control over tasks. It is not quite clear at this point what measures were used in previous work for determining the problem solvers' levels of confidence and whether claims were carefully studied. This issue merits further study.

## References


Anderson, J., & White, P. (2004). Problem solving in learning and teaching mathematics. In B. Perry, G. Anthony & C. Diezmann (Eds.), *Research in mathematics education in Australasia 2000-2003* (pp. 127-150). Flaxton, Qld: Post Pressed.

Cohors-Fressenborg, E., Kramer, S., Pundsack, F., Sjuts, J., & Sommer, N. (2010). The role of metacognitive monitoring in explaining differences in mathematics achievement. *ZDM Mathematics Education, 42*, 231-244.

Cohors-Fressenborg, E. , Sjuts, J., & Sommer, N. (2004). Komplexitt von Denkvorgngen und Formalisierung von Wissen. In M. Neubrand (Hrsg.), *Mathematische Kompetenzen von Shlerinnen und Shlern in Deutschland: Vertiefende Analysen im Rahmen von PISA-2000* (pp. 109-144). Wiesbaden: Verlag fr Sozialwissenschaften.

Demetriou, A. (2004). Mind intelligence and development: A cognitive, differential, and developmental theory of intelligence. In A. Dementriou & A. Raftopoulos (Eds.), *Developmental change: Theories, models and measurement* (pp. 21-73). Cambridge, UK: Cambridge University Press.

Elia, I., Heuvel–Panhuizen, M. V., & Kolovou, A. (2009). Exploring strategy use and strategy flexibility in non-routine problem solving by primary school high achievers in mathematics. *ZDM Mathematics Education, 41*, 605-618.

English, L. D. (2010). Modeling with complex data in the primary School. In R. Lesh, P. L. Galbraith, C. R. Haines, & A. Hurford (Eds.), *Modeling students' mathematical modeling competencies* (pp. 287-299).

English, L. D., Lesh, R. A., & Fennewald, T. (2008). *Future directions and perspectives for problem solving research and curriculum development*. Paper presented for TSG 19 at the International Congress on Mathematical Education. Monterrey, Mexico, July 6-13.

English, L. D., & Sriraman, B. (2010). Problem solving for the 21st century. In B. Sriraman & L. English (Eds.) *Theories of mathematics education* (pp. 263-290). New York: Springer Heidelberg.

Kantowski, M. G. (1977). Processes involved in mathematical problem solving. *Journal for Research in Mathematics Education, 8*, 163-180.

Lester, F. K. (1994). Musings about mathematical problem-solving research: 1970-1994. *Journal for Research in Mathematics Education, 25*(6), 660-675.

Mason, J., Burton, L. & Stacey, K. (1985). *Thinking mathematically*. London, Addison-Wesley.

Muir, T., Beswick, K., & Williamson, J. (2008). "I'm not very good at solving problems": An exploration of students' problem solving behaviors. *The Journal of Mathematical Behavior, 27*, 228-241.

Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 334-370). New York: Macmillan.